

Phase transition of generalized two dimensional Yang-Mills $U(N)$ on the sphere for $G(z) = z^4 + \lambda z^3$ and Maxwell construction

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The large- N behavior of the quartic-cubic generalized two dimensional Yang-Mills $U(N)$ on the sphere is investigated for finite cubic couplings. First, it is shown that there are two phase transitions one of which is third order and the other one is second order. Second, gYM_2 and Maxwell construction are compared and a relationship between two-dimensional space-time, that is purely mathematical, and four-dimensional space-time is obtained.

1 introduction

The YM_2 theory is defined by the lagrangian $tr(F^2)$ on a compact Riemann surface, where F is the 2-form field strength. If one considers $i tr(BF) + tr(B^2)$ as the Lagrangian of this theory, where B is an auxiliary pseudo-scalar field in the adjoint representation of the gauge group, and uses path-integral method over the field B , an effective Lagrangian of the form $tr(F^2)$ is concluded.

Because of two reasons, it is interesting that YM_2 theory is generalized. First, invariance under area-preserving diffeomorphisms and the lack of propagating degrees of freedom that are two important properties of YM_2 are not unique to the $i tr(BF) + tr(B^2)$ Lagrangian but it is possible to generalize the theory without losing these two properties. These generalized theories (gYM_2) are defined by replacing the $tr(B^2)$ term by an arbitrary class function $f(B)$ [5]. Second, it is conceivable that one of the generalized 2D models will reveal features which are more relevant and more closely resemble the four dimensional theories of interest.

Two dimensional Yang-Mills theory (YM_2) and generalized Yang-Mills theories (gYM_2 s) have been a subject of extensive study during recent years [1-18]. They are important theories because they are integrable. It was seen that there are certain relations between these theories and string theories. These relations can be seen from the study of the large N behavior of a YM_2 and gYM_2 based on a gauge group $SU(N)$ that is shown in [3,6,7,9]. On the other hand, these theories can shed light on some basic features of QCD_4 .

Because YM_2 and gYM_2 are integrable models so they are useful for exploring the general properties of QCD . For example, one can study the large- N behavior of the free energy of these theories. For this, one must start from the partition function of one of these theories on a certain surface. Then the sum

over reducible representations of $U(N)$ (or $SU(N)$), appearing in the expressions of partition function, must be replaced by a path integral over continuous Young tableaux, and calculated the area-dependence of the free energy from the saddle-point configuration. In [20], the logarithmic behavior of the free energy of $U(N)$ YM_2 on a sphere with area $A < A_c = \pi^2$ was obtained, and in [13] for $A > A_c$ it was proved that there is a third-order phase transition in YM_2 . A fact that was known earlier in the context of lattice formulation [21]. In the case of gYM_2 models, the same transition was shown for $G(z) = z^4$ in [18] and for $G(z) = z^6$ and $G(z) = z^2 + \lambda z^4$ in [22], and also for $G(z) = z^4 + \lambda z^3$ (for small λ) in [19], all on the sphere.

In [19], the large N behavior of a gYM_2 based on the gauge group $U(N)$ on the sphere was studied, for which $G(z) = z^4 + \lambda z^3$ and λ was small. It was seen that in that model, the density had two maximum with different heights. There was a transition when the absolute maximum exceeded one and also there was the other transition when the relative maximum exceeded one. So, there were two transitions both of which were third order.

In this paper the large N behavior of a gYM_2 based on a gauge group $U(N)$ on a sphere is also studied, for which $G(z) = z^4 + \lambda z^3$ and λ is not necessarily small. In Sec. 2, it is seen that there is a zeroth-order phase transition in free energy that is shown in next sections that is not correct. In Sec. 3, the zeroth-order phase transition is investigated by studying the density and it is obtained that there are four segments in each of which the density behaves differently. In Sec. 4, the segment 2) is studied and it is resulted that this segment must be removed. In Sec. 5, the segment 3) is studied and a third-order phase transition in free energy is obtained. In Sec. 6, the segment 3) is studied again and it is seen that there is the other phase transition that is second order. In Sec. 7, Maxwell construction is studied. In Sec. 8, gYM_2 and Maxwell construction are compared and it is seen that the main parameters in gYM_2 (that are purely mathematical parameters) are similar to physical parameters in the four-dimensional space-time. So, it is seen that there is a relationship between two-dimensional space-time and four-dimensional space-time. Section 9 is devoted to the concluding remarks.

2 The free energy

First, it is helpful to review the expression for the partition function of a gYM_2 on a sphere in the large N limit [18, 19]. The partition function of the gYM_2 on the sphere is [9, 10]

$$Z = \sum_r d_r^2 e^{-A\Lambda(r)}, \quad (1)$$

where r 's are the irreducible representations of the gauge group, d_r is the dimension of the r th representation, A is the area of the sphere and $\Lambda(r)$ is

$$\Lambda(r) = \sum_{k=1}^p \frac{a_k}{N^{k-1}} C_k(r), \quad (2)$$

in which C_k is the k th Casimir of group, and a_k 's are arbitrary constants. If one considers the gauge group $U(N)$ and parameterizes its representation by N integers $n_1 \geq n_2 \geq \dots \geq n_N$, it is seen that [23]

$$d_r = \prod_{1 \leq i \leq j \leq N} \left(1 + \frac{n_i - n_j}{j - i} \right),$$

$$C_k = \sum_{i=1}^N [(n_i + N - i)^k - (N - i)^k]. \quad (3)$$

For the partition function (1) be convergent, it is necessary that p in (2) be even and a_p be positive.

In the large N limit, the partition function (1) can be rewritten as a path integral over continuous parameters. If the continuous function $\phi(x)$ is introduced as

$$\phi(x) = -n(x) - 1 + x, \quad (4)$$

where

$$0 \leq x := i/N \leq 1, \quad (5)$$

and

$$n(x) := n_i/N, \quad (6)$$

then the partition function (1) becomes

$$Z = \int \prod_{0 \leq x \leq 1} d\phi(x) e^{S[\phi(x)]}, \quad (7)$$

where

$$S(\phi) = N^2 \left[-A \int_0^1 dx G[\phi(x)] + \int_0^1 \int_0^1 dy \log |\phi(x) - \phi(y)| \right], \quad (8)$$

apart from an unimportant constant, and

$$G(\phi) = \sum_{k=1}^p (-1)^k a_k \phi^k. \quad (9)$$

As $N \rightarrow \infty$, for determining the action (8), one should maximize S . The saddle point equation for S is

$$g[\phi(x)] = P \int_0^1 \frac{dy}{\phi(x) - \phi(y)}, \quad (10)$$

where

$$g(\phi) = \frac{A}{2} G'(\phi), \quad (11)$$

and P is the principal value of the integral. If the density is introduced as

$$\rho[\phi(x)] = \frac{dx}{d\phi(x)}, \quad (12)$$

then (10) becomes

$$g(z) = P \int_b^a \frac{\rho(\xi) d\xi}{z - \xi}, \quad (13)$$

and the normalization condition

$$\int_b^a \rho(\xi) d\xi = 1. \quad (14)$$

There is also another condition on the density, that is

$$0 \leq \rho(\xi) \leq 1. \quad (15)$$

The above condition can be obtained from the condition $n_1 \geq n_2 \geq \dots \geq n_N$. To solve (13), the function $H(z)$ is defined in the complex z -plane as [24]

$$H(z) := \int_b^a \frac{\rho(\xi) d\xi}{z - \xi}. \quad (16)$$

One can obtain [18]

$$H(z) = g(z) - \sqrt{(z-a)(z-b)} \sum_{m,n,q=0}^{\infty} \frac{(2n-1)!!(2q-1)!!}{2^{n+q}n!q!(n+q+m+1)!} a^n b^q z^m g^{(n+m+q+1)}(0), \quad (17)$$

and

$$\rho(z) = \frac{\sqrt{(a-z)(z-b)}}{\pi} \sum_{m,n,q=0}^{\infty} \frac{(2n-1)!!(2q-1)!!}{2^{n+q}n!q!(n+q+m+1)!} a^n b^q z^m g^{(n+m+q+1)}(0), \quad (18)$$

where $g^{(n)}$ is the n th derivative of g . Using (16) and (14), it is seen that $H(z)$ behaves like z^{-1} for large z . So, from (17), one can arrive at

$$\sum_{n,q=0}^{\infty} \frac{(2n-1)!!(2q-1)!!}{2^{n+q}n!q!(n+q)!} a^n b^q g^{(n+q)}(0) = 0, \quad (19)$$

$$\sum_{n,q=0}^{\infty} \frac{(2n-1)!!(2q-1)!!}{2^{n+q}n!q!(n+q-1)!} a^n b^q g^{(n+q-1)}(0) = 1. \quad (20)$$

These equations should be used to obtain a and b . Defining the free energy as

$$F := -\frac{1}{N^2} \ln Z, \quad (21)$$

one can obtain

$$F'(A) = \int_0^1 dx G[\phi(x)] = \int_b^a dz G(z) \rho(z), \quad (22)$$

that $F'(A)$ is the derivative of the free energy with respect to A . Using (9) and (22), it is seen that to obtain $F'(A)$, one should calculate the integrals

$$\int_b^a dz z^n \rho(z). \quad (23)$$

Using (16), by expanding $H(z)$ for $z \rightarrow \infty$, these integrals appear. So, using (17), one can calculate these integrals.

If one considers

$$G(z) = z^4 + \lambda z^3, \quad (24)$$

by the rescaling $\tilde{z} = z/\lambda$, one has

$$G(\tilde{z}) = \lambda^4 [\tilde{z}^4 + \tilde{z}^3]. \quad (25)$$

Using (12), one can obtain

$$\tilde{\rho}(\tilde{z}) = \lambda \rho(z) \quad (26)$$

where

$$\tilde{\rho}(\tilde{z}) := \frac{dx}{d\tilde{z}}. \quad (27)$$

Using (16), one has

$$\tilde{H}(\tilde{z}) = \lambda H(z) \quad (28)$$

where

$$\tilde{H}(\tilde{z}) := \int_{\tilde{b}}^{\tilde{a}} d\tilde{\xi} \frac{\tilde{\rho}(\tilde{\xi})}{\tilde{z} - \tilde{\xi}}. \quad (29)$$

Using (11) and (17), one obtains

$$\tilde{H}(\tilde{z}) = \tilde{A} \left[\frac{1}{2}(4\tilde{z}^3 + 3\tilde{z}^2) - \frac{1}{4}\sqrt{(\tilde{z} - \tilde{a})(\tilde{z} - \tilde{b})} \{3(\tilde{a} + \tilde{b} + 2\tilde{z}) + 3(\tilde{a}^2 + \tilde{b}^2) + 8\tilde{z}^2 + 2\tilde{a}\tilde{b} + 4\tilde{z}(\tilde{a} + \tilde{b})\} \right] \quad (30)$$

where $\tilde{A} = A \lambda^4$. Expanding (30) for large \tilde{z} , it is seen that

$$\tilde{H}(\tilde{z}) = \alpha_0 + \alpha_{-1}\tilde{z}^{-1} + O(\tilde{z}^{-2}). \quad (31)$$

As it was mentioned before, $\tilde{H}(\tilde{z})$ should behave like \tilde{z}^{-1} for $\tilde{z} \rightarrow \infty$. So using (30), one arrives at two following equations

$$\tilde{\tau}^2(3\tilde{\sigma} + \frac{3}{4}) + \tilde{\sigma}^2(2\tilde{\sigma} + \frac{3}{2}) = 0 \quad (32)$$

$$\frac{3}{4}\tilde{\tau}^4 + \tilde{\tau}^2(3\tilde{\sigma}^2 + \frac{3}{2}\tilde{\sigma}) = \frac{1}{\tilde{A}} \quad (33)$$

where

$$\tilde{\sigma} := \frac{\tilde{a} + \tilde{b}}{2}, \quad (34)$$

$$\tilde{\tau} := \frac{\tilde{a} - \tilde{b}}{2}. \quad (35)$$

Using (32) and (33), one obtains

$$\tilde{\tau}^2 = \tilde{\sigma}^2 \left(\frac{2\tilde{\sigma} + \frac{3}{2}}{3\tilde{\sigma} + \frac{3}{4}} \right), \quad (36)$$

$$\tilde{A} = -\frac{9\tilde{\sigma}^2 + \frac{9}{2}\tilde{\sigma} + \frac{9}{16}}{15\tilde{\sigma}^6 + \frac{45}{2}\tilde{\sigma}^5 + \frac{171}{16}\tilde{\sigma}^4 + \frac{27}{16}\tilde{\sigma}^3}. \quad (37)$$

Using (36) and the condition $\tilde{\tau}^2 \geq 0$, it is concluded that

$$-\frac{3}{4} \leq \tilde{\sigma} \leq -\frac{1}{4}.$$

Using (36), it is seen that for $\tilde{\sigma} = -\frac{1}{4}$, $\tilde{\tau}$ is infinity. Also, using (37), it is seen that for $\tilde{\sigma} = -\frac{3}{4}$, \tilde{A} is infinity. So the condition on $\tilde{\sigma}$ is converted to

$$-\frac{3}{4} < \tilde{\sigma} < -\frac{1}{4}. \quad (38)$$

If one expands (16) for large \tilde{z} and uses (14), the derivative of the free energy with respect to the area, using (22) and (24), becomes

$$\tilde{F}'(\tilde{A}) = \tilde{H}_4(\tilde{z}) + \tilde{H}_5(\tilde{z}), \quad (39)$$

where $\tilde{H}_4(\tilde{z})$ (or $\tilde{H}_5(\tilde{z})$) is the coefficient of \tilde{z}^{-4} (or \tilde{z}^{-5}) in the expansion of $\tilde{H}(\tilde{z})$. So, expanding (30) for large \tilde{z} and using (37), it is seen that

$$\tilde{F}'(\tilde{A}) = -\frac{\tilde{\sigma}^3(36 + 390\tilde{\sigma} + 1667\tilde{\sigma}^2 + 3500\tilde{\sigma}^3 + 3600\tilde{\sigma}^4 + 1600\tilde{\sigma}^5)}{12(1 + 4\tilde{\sigma})^2(3 + 15\tilde{\sigma} + 20\tilde{\sigma}^2)}. \quad (40)$$

Integrating (40) and using (37), regardless a constant, leads to

$$\tilde{F}(\tilde{\sigma}) = \frac{1}{4} \left[-\frac{4}{3 + 4\tilde{\sigma}} + \frac{3 + 8\tilde{\sigma}}{2(3 + 15\tilde{\sigma} + 20\tilde{\sigma}^2)^2} + \frac{5(1 + 4\tilde{\sigma})}{3 + 15\tilde{\sigma} + 20\tilde{\sigma}^2} - 4 \log |\tilde{\sigma}| + 2 \log |1 + 4\tilde{\sigma}| - 2 \log |3 + 4\tilde{\sigma}| \right]. \quad (41)$$

Using (41) and (37), one can plot \tilde{F} as a function of \tilde{A} that is shown in Fig. 2 and also using (40) and (37), one can plot \tilde{F}' as a function of \tilde{A} that is shown in Fig. 3. Because for each area the free energy must be minimum, so using Fig. 2, it is concluded that there is a zeroth-order phase transition in the free energy of the system. Now the question is whether the zeroth-order phase transition is a correct result? In the next section, the question will be answered.

3 The boundary conditions on density

Using (37), one can plot \tilde{A} as a function of $\tilde{\sigma}$, that is shown in Fig. 1. In this figure, it is seen that in the interval $\tilde{A}_I \leq \tilde{A} \leq \tilde{A}_{II}$, for each \tilde{A} , there are two or three values for $\tilde{\sigma}$. So, using (41), for an area there are several \tilde{F} in this interval. Now, one should study the density of the system and investigate the condition (15). So, using (11) and (18), the density for $\tilde{G}(\tilde{z}) = \tilde{z}^4 + \tilde{z}^3$ is

$$\tilde{\rho}(\tilde{z}) = \frac{\tilde{A}}{2\pi} \sqrt{\tilde{\tau}^2 - (\tilde{z} - \tilde{\sigma})^2} [4\tilde{\sigma}^2 + 2\tilde{\tau}^2 + 4\tilde{\sigma}\tilde{z} + 4\tilde{z}^2 + 3(\tilde{\sigma} + \tilde{z})]. \quad (42)$$

By differentiating $\tilde{\rho}$ with respect to \tilde{z} and putting it equal to zero, one obtains

$$6(1 + 4\tilde{\sigma})\tilde{z}^3 + \{3 + 6\tilde{\sigma}(1 - 4\tilde{\sigma})\}\tilde{z}^2 + \tilde{\sigma}(-3 - 6\tilde{\sigma} + 8\tilde{\sigma}^2)\tilde{z} + \tilde{\sigma}^2(1 + 2\tilde{\sigma})(3 + 4\tilde{\sigma}) = 0. \quad (43)$$

This equation has three roots for which the density is extremum. One can plot the roots of the equation and \tilde{b} and \tilde{a} (using (34), (35) and (36)) as a function of $\tilde{\sigma}$ in the interval $-\frac{3}{4} < \tilde{\sigma} < -\frac{1}{4}$. This is shown in Fig. 4. The density is acceptable only for $\tilde{b} \leq \tilde{z} \leq \tilde{a}$, because it has been defined just in the interval. So, using Fig. 4, it is seen that there are three intervals in each of which the density behaves differently. This means that one should plot the density in each interval, separately. These are shown in figures 5, 6 and 7. Each of these figures are related to specific $\tilde{\sigma}$'s. When $\tilde{\sigma}$ is changed, the curves in Fig. 5 and Fig. 6 change in terms of quantity, but don't change in terms of quality. For some $\tilde{\sigma}$'s, Fig. 7 is converted to Fig. 8. One can plot $\tilde{\rho}_i$'s as a function of $\tilde{\sigma}$, that $\tilde{\rho}_i$'s are the extremum of $\tilde{\rho}$ with respect to \tilde{z} . This is shown in Fig. 9. Using Fig. 9, the boundary of Fig. 7 and Fig. 8 is obtained. Using Fig. 4 and Fig 9, it is seen that there are four intervals for $\tilde{\sigma}$ as follows: 1) $-\frac{3}{4} < \tilde{\sigma} \leq \tilde{\sigma}_I$ that is related to Fig. 5, 2) $\tilde{\sigma}_I \leq \tilde{\sigma} \leq \tilde{\sigma}_{II}$ that is related to Fig. 6, 3) $\tilde{\sigma}_{II} \leq \tilde{\sigma} \leq \tilde{\sigma}_{III}$ that is related to Fig. 7 and 4) $\tilde{\sigma}_{III} \leq \tilde{\sigma} < -\frac{1}{4}$ that is related to Fig. 8. The condition (using (26) and (15))

$$0 \leq \tilde{\rho} \leq \lambda \quad (44)$$

restricts the acceptable densities. So, one should redefine the density in the intervals 2) and 3), because in these intervals the density is not nonnegative, and then find the free energies of these redefined densities. So, the free energy of the system is changed. As a result, the zeroth-order phase transition in the free energy obtained in previous section is incorrect. In the next three sections, the density in two intervals 2) and 3) will be redefined and the order of the phase transition in the free energy will be obtained.

4 studying of the second interval

One can obtain $\tilde{\sigma}_I = -0.601986$ and $\tilde{\sigma}_{II} = -0.418476$, and also $\tilde{A}_I = 37.8042$ and $\tilde{A}_{II} = 62.2248$. In Fig. 6 it is clear that in this interval the density is not nonnegative, so it must be redefined. If one redefines it as

$$\tilde{\rho}_2(\tilde{z}) = \begin{cases} 0, & \tilde{z} \in [\tilde{c}, \tilde{a}] \\ \tilde{\rho}_2(\tilde{z}), & \tilde{z} \in [\tilde{b}, \tilde{c}] \end{cases} \quad (45)$$

and finds $\tilde{\rho}_2(\tilde{z})$ as a function of $\tilde{\delta}$ where $\tilde{\delta} = \frac{\tilde{b}+\tilde{c}}{2}$, it is seen that $\tilde{\rho}_2(\tilde{z})$ (as a function of $\tilde{\delta}$) is the same as $\tilde{\rho}_2(\tilde{z})$ (as a function of $\tilde{\sigma}$). So the behavior of $\tilde{\rho}_2(\tilde{z})$ is similar to that of $\tilde{\rho}_2(\tilde{z})$ but in the interval $\tilde{b} \leq \tilde{z} \leq \tilde{c}$. This means that for some \tilde{z} 's, $\tilde{\rho}_2(\tilde{z})$ is negative. So this redefining of the density is incorrect. If it is redefined as

$$\tilde{\rho}_2(\tilde{z}) = \begin{cases} 0, & \tilde{z} \in [\tilde{d}, \tilde{e}] \\ \tilde{\rho}_2(\tilde{z}), & \tilde{z} \in [\tilde{b}, \tilde{d}] \cup [\tilde{e}, \tilde{a}] \end{cases} \quad (46)$$

one can find [18]

$$\tilde{H}_2(\tilde{z}) = \frac{\tilde{A}}{2} \left[(4\tilde{z}^3 + 3\tilde{z}^2) - \sqrt{(\tilde{z} - \tilde{b})(\tilde{z} - \tilde{d})(\tilde{z} - \tilde{e})(\tilde{z} - \tilde{a})} \{3 + 4\tilde{z} + 2(\tilde{a} + \tilde{b} + \tilde{d} + \tilde{e})\} \right], \quad (47)$$

and

$$\tilde{\rho}_2(\tilde{z}) = \frac{\tilde{A}}{2\pi} \sqrt{(\tilde{z} - \tilde{b})(\tilde{z} - \tilde{d})(\tilde{z} - \tilde{e})(\tilde{a} - \tilde{z})} \{3 + 4\tilde{z} + 2(\tilde{a} + \tilde{b} + \tilde{d} + \tilde{e})\}. \quad (48)$$

Expanding $\tilde{H}_2(\tilde{z})$ for large \tilde{z} , it is seen that

$$\tilde{H}_2(\tilde{z}) = \beta_1 \tilde{z} + \beta_0 + \beta_{-1} \tilde{z}^{-1} + O(\tilde{z}^{-2}). \quad (49)$$

Because $\tilde{H}_2(\tilde{z})$ should behave like \tilde{z}^{-1} , for large \tilde{z} , so one arrives at three following equations

$$\beta_1 = 0, \quad (50)$$

$$\beta_0 = 0, \quad (51)$$

$$\beta_{-1} = 1. \quad (52)$$

One also has [18]

$$\int_{\tilde{d}}^{\tilde{e}} d\tilde{z} \{g(\tilde{z}) - \tilde{H}_2(\tilde{z})\} = 0, \quad (53)$$

where $g(\tilde{z}) = \frac{\tilde{A}}{2} G'(\tilde{z})$. So one can obtain

$$\int_{\tilde{d}}^{\tilde{e}} d\tilde{z} \left[\frac{\tilde{A}}{2} \sqrt{(\tilde{z} - \tilde{b})(\tilde{z} - \tilde{d})(\tilde{z} - \tilde{e})(\tilde{a} - \tilde{z})} \{3 + 4\tilde{z} + 2(\tilde{a} + \tilde{b} + \tilde{d} + \tilde{e})\} \right] = 0. \quad (54)$$

Using this equation and (50), (51) and (52), one can obtain the four unknowns \tilde{a} , \tilde{b} , \tilde{e} and \tilde{d} . To study the structure of the phase transition, one can use the following change of variables

$$\tilde{b} = \tilde{b}_c(1 + P), \quad \tilde{d} = \tilde{d}_c(1 + M), \quad \tilde{e} = \tilde{e}_c(1 + X), \quad \tilde{a} = \tilde{a}_c(1 + U), \quad (55)$$

where the index c shows the critical point between intervals 1) and 2), that is marked with the symbol I in Fig. 1. By calculation, it is seen that

$$\tilde{b}_c = -0.920721, \quad \tilde{d}_c = \tilde{e}_c = \tilde{a}_c = -0.283251 \quad (56)$$

One can substitute (55) and (56) to (50) and (51), and obtain

$$P = -0.12M - 0.06M^2 - 0.12U - 0.06U^2 - 0.12X - 0.06X^2 - 0.04MU - 0.04UX - 0.04MX, \quad (57)$$

$$U = -M - 0.232M^2 - X - 0.232X^2 - 0.232MX. \quad (58)$$

Using these two relations, it can be seen that if $M = -X$ then P, U, X^2 and M^2 are of the same order but if $M \neq -X$ then P, M, X and U are of the same order. If $M = -X$, using (55) and (56), it is seen that $\tilde{a} < \tilde{e}$. This is an incorrect result, because for \tilde{b} , \tilde{d} , \tilde{e} and \tilde{a} , there is the following condition (using (46))

$$\tilde{b} < \tilde{d} < \tilde{e} < \tilde{a}, \quad (59)$$

so $M \neq -X$. One can consider

$$\begin{aligned} P &= P_{1/2} \Omega^{1/2} + P_1 \Omega + P_{3/2} \Omega^{3/2} + P_2 \Omega^2, \\ M &= M_{1/2} \Omega^{1/2} + M_1 \Omega + M_{3/2} \Omega^{3/2} + M_2 \Omega^2, \\ X &= X_{1/2} \Omega^{1/2} + X_1 \Omega + X_{3/2} \Omega^{3/2} + X_2 \Omega^2, \\ U &= U_{1/2} \Omega^{1/2} + U_1 \Omega + U_{3/2} \Omega^{3/2} + U_2 \Omega^2, \end{aligned} \quad (60)$$

where Ω is $(\tilde{A} - \tilde{A}_c)/\tilde{A}_c$ and $\tilde{A}_c = \tilde{A}_I$. From (55), (56) and (60), and using (50), (51), (52) and (54), one can obtain

$$\begin{aligned} P &= -0.067 \Omega + 0.0127 \Omega^{3/2} + 0.05 \Omega^2, \\ M &= 0.64 \Omega^{1/2} - 0.064 \Omega - 0.266 \Omega^{3/2} + X_2 \Omega^2, \\ X &= 0.64 \Omega^{1/2} - 0.064 \Omega - 0.266 \Omega^{3/2} + X_2 \Omega^2, \\ U &= -1.28 \Omega^{1/2} - 0.16 \Omega + 0.34 \Omega^{3/2} + (0.038 - 2X_2) \Omega^2. \end{aligned} \quad (61)$$

Using (61), (55), (56), (48) and $\tilde{A} = \tilde{A}_I(1 + \Omega)$, one can plot the density as a function of \tilde{z} for small Ω . It is seen that for $\Omega \neq 0$, this graph is similar to Fig. 6. So the redefined density is not also nonnegative. It follows that the interval 2) must not exist and the curve between I and II, in figures 1 to 3, must be removed.

5 The third-order phase transition

The interval 3) is $\tilde{\sigma}_{II} \leq \tilde{\sigma} \leq \tilde{\sigma}_{III}$. One can obtain $\tilde{\sigma}_{II} = -0.418476$ and $\tilde{\sigma}_{III} = -0.33541$, and also $\tilde{A}_{II} = 62.2248$ and $\tilde{A}_{III} = 25.5689$. From Fig. 7 it is seen that the density is not nonnegative in this interval, and so it must be redefined. In line with the previous section, if one redefines the density as

$$\tilde{\rho}_3(\tilde{z}) = \begin{cases} 0, & \tilde{z} \in [\tilde{d}, \tilde{e}] \\ \tilde{\rho}_3(\tilde{z}), & \tilde{z} \in [\tilde{b}, \tilde{d}] \cup [\tilde{e}, \tilde{a}] \end{cases} \quad (62)$$

it can be concluded that relations (47) to (54) in the interval 2) also exist in the interval 3). To study the structure of the phase transition at point *III*, one can use the following change of variables

$$\tilde{b} = \tilde{b}_{III}(1 + P), \quad \tilde{d} = \tilde{d}_{III}(1 + M), \quad \tilde{e} = \tilde{e}_{III}(1 + X), \quad \tilde{a} = \tilde{a}_{III}(1 + U), \quad (63)$$

where

$$\tilde{b}_{III} = -0.938782, \quad \tilde{a}_{III} = 0.267962, \quad \tilde{d}_{III} = \tilde{e}_{III} = -0.207295 \quad (64)$$

Substituting (63) and (64) for the parameters in (50) and (51), it is seen that

$$\begin{aligned} P &= -0.047 M^2 - 0.18 U - 0.03 M X - 0.047 X^2 \\ U &= -0.15 M^2 - 0.1 M X - 0.15 X^2 \end{aligned} \quad (65)$$

Using these two relations, it is clear that P , U , M^2 and X^2 are of the same order. So one can consider

$$\begin{aligned} P &= P_1 \Omega + P_{3/2} \Omega^{3/2} + P_2 \Omega^2, \\ M &= M_{1/2} \Omega^{1/2} + M_1 \Omega + M_{3/2} \Omega^{3/2} + M_2 \Omega^2, \\ X &= X_{1/2} \Omega^{1/2} + X_1 \Omega + X_{3/2} \Omega^{3/2} + X_2 \Omega^2, \\ U &= U_1 \Omega + U_{3/2} \Omega^{3/2} + U_2 \Omega^2, \end{aligned} \quad (66)$$

where $\Omega = \frac{\tilde{A} - \tilde{A}_{III}}{\tilde{A}_{III}}$. From (63), (64) and (66), and using (50), (51), (52) and (54), it is obtained

$$\begin{aligned} P &= -0.064 \Omega + 0.048 \Omega^2, \\ M &= 1.61 \Omega^{1/2} - 0.125 \Omega - 0.498 \Omega^{3/2} + (-0.01 - X_2) \Omega^2, \\ X &= -1.61 \Omega^{1/2} - 0.125 \Omega + 0.498 \Omega^{3/2} + X_2 \Omega^2, \\ U &= -0.535 \Omega + 0.18 \Omega^2. \end{aligned} \quad (67)$$

Using (63), (64), (67), (48) and $\tilde{A} = \tilde{A}_{III} (1 + \Omega)$, one can plot the density as a function of \tilde{z} for small Ω , that is shown in Fig. 10. It is clear that for $\Omega \neq 0$, the density is nonnegative and so the redefining (62) in the interval 3) around

point *III* is correct. Now one can obtain the phase transition in the free energy around point *III*. Expanding (47) for large \tilde{z} and using (39), one can obtain

$$(\tilde{F}'_3)_{III}(\tilde{A}) = -0.07 - 0.034\Omega + 0.0129\Omega^2 + O(\Omega^3). \quad (68)$$

To calculate the phase transition, one should find $\tilde{F}'_w(\tilde{A})$ around point *III*. This is the derivative of the free energy obtained using the initial density (not the redefined density). Replacing $\tilde{\sigma} = \Phi + \tilde{\sigma}_{III}$ in (37) and using $\tilde{A} = \tilde{A}_{III}(1 + \Omega)$, one can obtain

$$\Phi = -0.0414\Omega + 0.00747\Omega^2. \quad (69)$$

So using (40), it is obtained

$$(\tilde{F}'_w)_{III}(\tilde{A}) = -0.07 - 0.034\Omega + 0.0066\Omega^2 + O(\Omega^3). \quad (70)$$

Using (68) and (70), one can obtain

$$(\tilde{F}'_3)_{III}(\tilde{A}) - (\tilde{F}'_w)_{III}(\tilde{A}) = 0.0063\Omega^2 + O(\Omega^3). \quad (71)$$

As a result, there is a third-order phase transition in the free energy around point *III*.

6 The second-order phase transition

In Fig. 1 if one moves on the graph from the right side to the left passing through the point *III*, Fig. 8 is converted to Fig. 7. Redefining the density in the interval 3) similar to section 5 (relation (62)), Fig. 7 will be replaced by Fig. 10. If \tilde{A} increases continuously, the distance between \tilde{e} and \tilde{a} in Fig. 10 goes to zero. Thus in Fig. 1, the curve between points *III* and *II* will be replaced by the curve between points *III* and *IV* shown in Fig. 11. As a result, in Fig. 11, the curves between points *III* and *II* and between points *II* and *I* must be removed and using the fact that for each area there is specific density, the curve between *I* and *V* must be removed too. This is due to the fact that in the interval $\tilde{\sigma}_V < \tilde{\sigma} < \tilde{\sigma}_I$, the graph of the density as a function of \tilde{z} is similar to Fig. 5 while in the interval $\tilde{\sigma}_{IV} < \tilde{\sigma} < \tilde{\sigma}_{III}$, it is similar to Fig. 10. In this section, first, points *IV* and *V* will be found and then the phase transition in the free energy, to go from *IV* to *V*, will be obtained.

At point *IV*, \tilde{a} and \tilde{e} are equal, because the density at point *IV* is the same as one at point *V*. So one can use the following change of variables

$$\tilde{a} = \tilde{e} = \nu, \quad \tilde{d} = \gamma + \eta, \quad \tilde{b} = \gamma - \eta. \quad (72)$$

Using (50), one obtains

$$\eta^2 = -\frac{4\gamma^2 + (\gamma + \nu)(3 + 4\nu)}{2}. \quad (73)$$

Now, using (51), (52) and the above relation, the following two equations are obtained

$$-2\gamma\nu(3+4\gamma+4\nu) - \frac{1}{2}(3+8\gamma)\{4\gamma^2 + (\gamma+\nu)(3+4\nu)\} = 0, \quad (74)$$

$$-\frac{3}{16}\{\gamma+4\gamma^2-4\gamma\nu-\nu(3+4\nu)\}\{4\gamma^2+(\gamma+\nu)(3+4\nu)\}\tilde{A} = 1. \quad (75)$$

The roots of the equation (74) are

$$\nu_1 = -\frac{9+48(\gamma+\gamma^2) + \sqrt{81+432\gamma-288\gamma^2-3840\gamma^3-3840\gamma^4}}{24+96\gamma} \quad (76)$$

and

$$\nu_2 = -\frac{9+48(\gamma+\gamma^2) - \sqrt{81+432\gamma-288\gamma^2-3840\gamma^3-3840\gamma^4}}{24+96\gamma}. \quad (77)$$

Using (73) and (54), one can obtain

$$\int_{\gamma+\eta}^{\nu} d\tilde{z} \left[\frac{\tilde{A}}{2} (3+4\tilde{z}+4\gamma+4\nu)(\tilde{z}-\nu) \sqrt{\tilde{z}^2 - 2\tilde{z}\gamma + \frac{1}{2}\{6\gamma^2 + (\gamma+\nu)(3+4\nu)\}} \right] = 0. \quad (78)$$

Now, one should calculate this integral and substitute (76) and (77) for ν in the result of the integral, consecutively. Using ν_1 , one can obtain

$$(\gamma_1 = -0.67748, \tilde{A}_1 = 48.2053), \quad (\gamma_2 = -0.42, \tilde{A}_2 = 62.2), \quad (79)$$

and using ν_2 , one also can obtain

$$\gamma_3 = -0.602, \quad \tilde{A}_3 = 37.8042, \quad (80)$$

that (75) has been used to obtain \tilde{A}_i 's. It is seen that ν_1 , γ_2 , and \tilde{A}_2 are related to point *II* and so γ_2 is not an acceptable answer, because at this point the diagram of the density is like Fig. 6 that is not nonnegative. Also it is seen that ν_2 , γ_3 and \tilde{A}_3 are related to point *I*. By plotting the density related to \tilde{A}_1 and \tilde{A}_3 separately, it is seen both of which are like Fig. 5, and so both of the areas \tilde{A}_1 and \tilde{A}_3 are apparently correct answers. As a result, one should find point *IV* numerically and investigate that whether the correct answer is \tilde{A}_1 or \tilde{A}_3 . Using (50), (51), (52) and (54) and starting from point *III*, by increasing \tilde{A} little by little, one can obtain for $\tilde{A}_I = \tilde{A}_3$, using the relations in the interval 3), the values $\tilde{b} = -0.9069$, $\tilde{d} = -0.4689$, $\tilde{e} = -0.09765$ and $\tilde{a} = 0.2117$. So, one can plot the density related to \tilde{A}_I as a function of \tilde{z} . It is seen that the graph is like Fig. 10. So \tilde{A}_3 is not related to point *IV*, because at this point the graph of the density is like Fig. 5. Increasing \tilde{A} further, one can plot the density related to \tilde{A}_1 as a function of \tilde{z} . It is seen that the diagram is like Fig. 5, and so point *IV* is specified by \tilde{A}_1 , γ_1 , and ν_1 . So the numerical values of the unknown parameters related to point *IV* are as follows

$$\tilde{A}_{IV} = 48.2053, \quad \tilde{a}_{IV} = \tilde{e}_{IV} = 0.12023, \quad \tilde{d}_{IV} = -0.44966, \quad \tilde{b}_{IV} = -0.905313 \quad (81)$$

Using

$$\tilde{b} = \tilde{b}_{IV}(1 + P), \quad \tilde{d} = \tilde{d}_{IV}(1 + M), \quad \tilde{e} = \tilde{e}_{IV}(1 + X), \quad \tilde{a} = \tilde{a}_{IV}(1 + U) \quad (82)$$

and also using (50) and (51), one can obtain

$$\begin{aligned} P &= 0.0729U + 0.0729X + 0.03699U^2 + 0.03699X^2 + 0.0306UX, \\ M &= -5.584P - 0.3257U - 0.3257X - 0.0938U^2 - 0.0938X^2 - 0.0626UX. \end{aligned} \quad (83)$$

Using these two relations, it is clear that if $U = -X$ then P , M , U^2 and X^2 are of the same order, but if $U \neq -X$ then P , M , U and X are of the same order.

So for $U \neq -X$, one can consider

$$\begin{aligned} P &= P_1 \Omega + P_2 \Omega^2 \\ M &= M_1 \Omega + M_2 \Omega^2 \\ U &= U_1 \Omega + U_2 \Omega^2 \\ X &= X_1 \Omega + X_2 \Omega^2, \end{aligned} \quad (84)$$

and use (50), (51), (52) and (54), and obtain the unknown parameters P_1 to X_2 . It is seen that there are two sets of numerical values for P_1 to X_2 . Using one of these sets, it is seen that $X = -0.9375\Omega$, and $U = -0.0057\Omega$. Using (81) and (82), $\tilde{a} < \tilde{e}$ is obtained that is incorrect. Using the other one, it is seen that $\tilde{e} = \tilde{a}$, up to $O(\Omega^2)$. So this is also incorrect. Thus $U \neq -X$ is not acceptable, and surely $U = -X$.

For $U = -X$, one can expand (50) and (51) up to order P (so up to order M , X^2 and U^2) and obtain

$$P = 0.04X^2, \quad M = -0.36X^2. \quad (85)$$

Using (85), (52) and $\tilde{A} = \tilde{A}_{IV}(1 + \Omega)$, it is obtained (up to order $\sqrt{\Omega}$)

$$(X)_1 = -1.15i\sqrt{\Omega}, \quad (X)_2 = 1.15i\sqrt{\Omega}. \quad (86)$$

Because Ω is negative, and $U = -X$, so the correct answer (up to order $\sqrt{\Omega}$) is $X = 1.15i\sqrt{\Omega}$. Thus one can obtain

$$X = 1.15i\sqrt{\Omega} + X_1\Omega, \quad U = -1.15i\sqrt{\Omega} - X_1\Omega, \quad P = -0.057\Omega, \quad M = 0.487\Omega. \quad (87)$$

Expanding $\tilde{H}(\tilde{z})$ for large \tilde{z} , as before, and then using $\tilde{F}'_3(\tilde{A}) = \tilde{H}_4(\tilde{z}) + \tilde{H}_5(\tilde{z})$ around point IV , it is seen that

$$(\tilde{F}'_3)_{IV}(\tilde{A}) = -0.09396 - 0.031046\Omega + O(\Omega^2). \quad (88)$$

Now, one should find point V that is in the interval 1) and $\tilde{A}_V = \tilde{A}_{IV} = 48.2053$. Because the areas related to the points V and IV are the same, so the graph of

the density in these two points must be the same in terms of both quality and quantity. It can be concluded that

$$\tilde{a}_V = \tilde{d}_{IV} = -0.44966, \quad \tilde{b}_V = \tilde{b}_{IV} = -0.905313, \quad \tilde{\sigma}_V = -0.6775 \quad (89)$$

So one can obtain \tilde{F}'_w around point V , like section 5. Up to order Ω , one can arrive at

$$(\tilde{F}'_w)_V(\tilde{A}) = -0.09396 - 0.013411 \Omega + O(\Omega^2). \quad (90)$$

Using (88) and (90), it is seen that

$$(\tilde{F}'_3)_{IV}(\tilde{A}) - (\tilde{F}'_w)_V(\tilde{A}) = -0.017635 \Omega + O(\Omega^2). \quad (91)$$

It is clear that there is a second-order phase transition in the free energy and $(\tilde{F}'_3)_{IV}(\tilde{A}) > (\tilde{F}'_w)_V(\tilde{A})$, because Ω is negative. Now, one can plot \tilde{F}' as a function of \tilde{A} that is shown in Fig. 12. In this figure, the paths from I to II , I to V , and II to III are wrong paths and must be removed and the directed path from III to IV is correct. Meanwhile, in Fig. 12 there are two regions resembling triangles, and the surface area of these two triangles should be the same. But the proof for that comes from the fact that one can go from point III to point IV through two different paths, the wrong one which goes through the lower curve (III to II , II to I , and then I to IV), and the correct one which goes from III to IV directly. The difference of the free energies at IV and III should be the same following both paths. This results that the surface areas of the triangle-like regions are equal.

7 Maxwell construction

Van der Waals equation of state for a gas is

$$(p + \frac{an^2}{V^2})(V - nb) = nRT, \quad (92)$$

where n is the number of moles, T is temperature, p is pressure, and V is the total volume of the gas. R is the gas constant that is $R = 8.3145 \frac{J}{mol.K}$. a and b are positive experimental constants that are specific for each gas. This relation, can also be rewritten as

$$(p + \frac{a'}{v^2})(v - b') = k_B T, \quad (93)$$

where $b' = \frac{b}{N_A}$, $a' = \frac{a}{N_A^2}$, $v = \frac{V}{N}$ (N is the total number of particles), $k_B = 1.38 \times 10^{-23} \frac{J}{K}$ that is Boltzmann's constant, and $N_A = 6.02 \times 10^{23}$ that is Avogadro's number. Using above relation, it is seen that in inflection point, using

$$\frac{\partial p}{\partial v}|_c = 0, \quad \frac{\partial^2 p}{\partial v^2}|_c = 0, \quad (94)$$

there are the following three conditions

$$v_c = 3b', \quad T_c = \frac{8a'}{27b'k_B}, \quad p_c = \frac{a'}{27b'^2}. \quad (95)$$

The point with these conditions is named critical point. If (93) is rewritten in terms of \tilde{v} , \tilde{p} and \tilde{T} where

$$\tilde{v} = \frac{v}{v_c}, \quad \tilde{p} = \frac{p}{p_c}, \quad \tilde{T} = \frac{T}{T_c}, \quad (96)$$

then (93) is converted to

$$(3\tilde{v} - 1)(\tilde{p} + \frac{3}{\tilde{v}^2}) = 8\tilde{T}. \quad (97)$$

It is seen that the above relation is independent of a' and b' . Now, one can plot \tilde{p} as a function of \tilde{v} (for van der Waals isotherms), that is shown in Fig.13, for $T = T_c$, $T > T_c$, and $T < T_c$. By calculating and using the principle of the least energy, it can be seen that for $T < T_c$, the curve of isotherm is converted to Fig. 14. In this figure, the curve between 1 and 2 should be removed and 1 must be connected to 2 directly and also the surface areas B and C should be equal to each other[25]. If one plots chemical potential as a function of pressure, that is shown in Fig. 15, compared with Fig. 14, the region resembling triangle should be removed from the graph. So, there is a phase transition of first order in chemical potential in point 1 (or 2) and as a result there is a phase transition of second order in free energy of the substance, because chemical potential is derivative of the free energy.

8 Comparing gYM_2 with Maxwell construction

Comparing figures 11 and 14 and also figures 12 and 15, it is seen that \tilde{A} is equivalent to pressure, $\tilde{\sigma}$ is equivalent to volume and \tilde{F}' is equivalent to chemical potential. The points V and IV in gYM_2 are equivalent to the points 1 and 2 in Maxwell construction, respectively. To go from V to IV there is a second-order phase transition in the free energy of the system, and also the same transition happens to go from 1 to 2. In Maxwell construction, the surface areas B and C are equal to each other (in Fig. 14). In gYM_2 , in Fig. 12, the surface areas of the triangle-like regions are equal.

If one considers $G(z) = z^4 + \lambda z^2$ instead of $G(z) = z^4 + \lambda z^3$, and uses the relations of section 2, it is obtained

$$\tilde{A} = \frac{12}{-1 + 4\tilde{\sigma}^4}, \quad (98)$$

where $\tilde{A} = A\lambda^2$ and $\tilde{\sigma} = \frac{\sigma}{\sqrt{\lambda}}$. Now, \tilde{A} can be plotted as a function of $\tilde{\sigma}$ that is shown in Fig. 16. This figure is similar to Fig. 13 for $T > T_c$.

Also, if one considers $G(z) = z^4 + \lambda z$ instead of $G(z) = z^4 + \lambda z^3$, and uses section 2, it is obtained

$$\tilde{A} = \frac{48\tilde{\sigma}^2}{1 - 16\tilde{\sigma}^3 - 80\tilde{\sigma}^6}, \quad (99)$$

where $\tilde{A} = A\lambda^{4/3}$ and $\tilde{\sigma} = \frac{\sigma}{\lambda^{1/3}}$. So, one can plot \tilde{A} as a function of $\tilde{\sigma}$ that is shown in Fig. 17. This figure is also similar to Fig. 13 for $T > T_c$.

As a result, in gYM_2 , the model $G(z)$ plays the role of temperature.

9 Concluding remarks

A gYM_2 with quartic and cubic couplings was studied. The effect of the cubic coupling on the density and the free energy was investigated. It was seen that there were four intervals that in two of which the density was not nonnegative and so had to be redefined. Redefining the density caused two phase transitions in the free energy, one of which was third order and the other one was second order. In the end, gYM_2 and Maxwell construction were compared and it was seen that there was a relationship between two-dimensional space-time, that is purely mathematical, and four-dimensional space-time.

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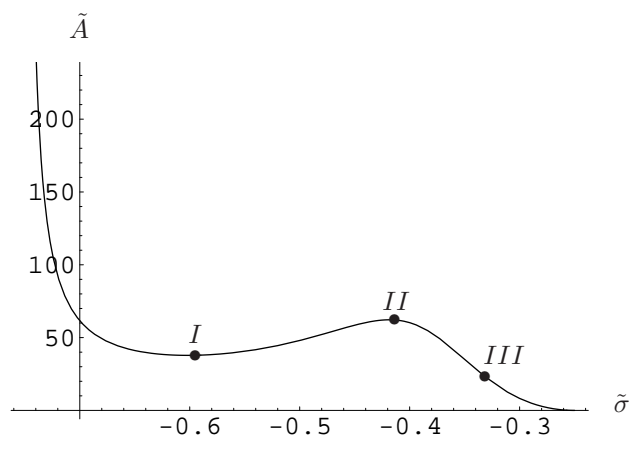


Figure 1: The surface area

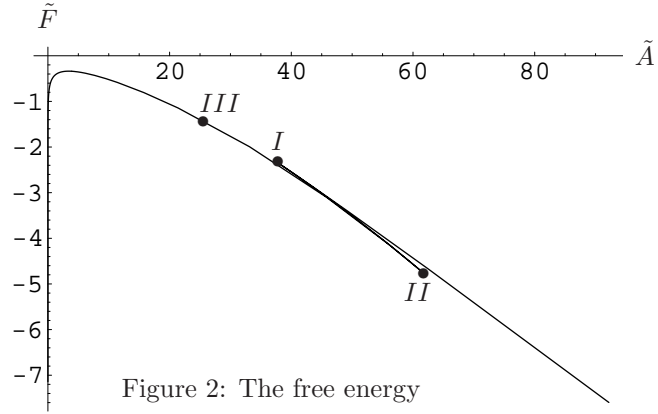


Figure 2: The free energy

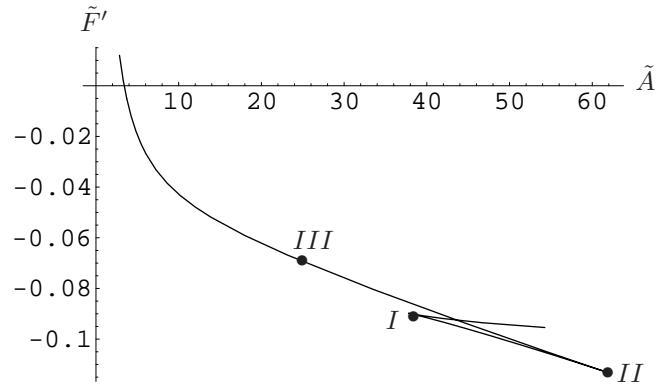


Figure 3: The derivative of the free energy

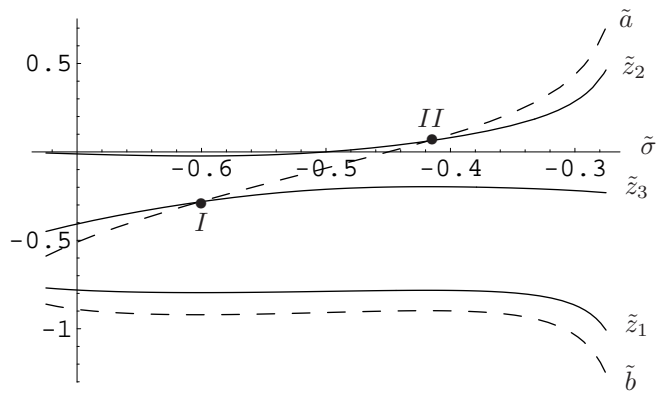


Figure 4: The roots of the density

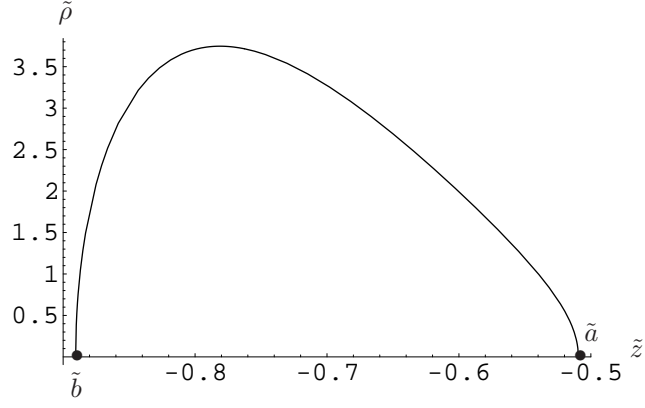


Figure 5: The density in interval 1)

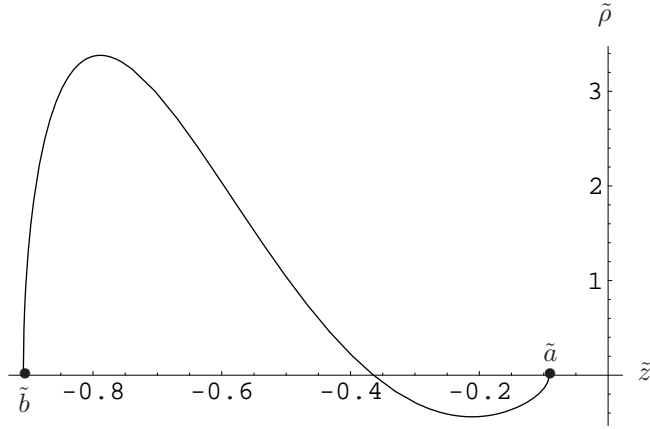


Figure 6: The density in interval 2)

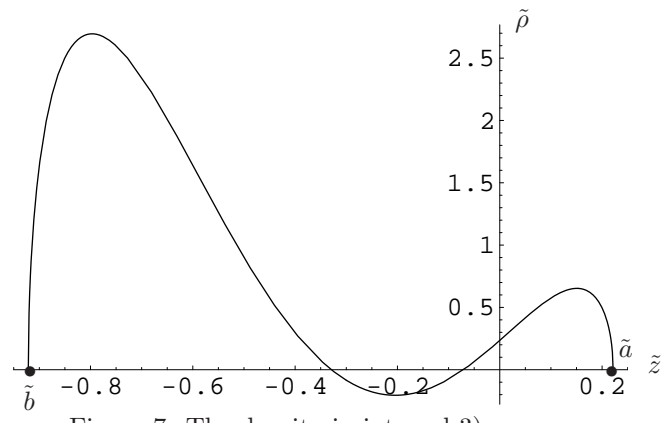


Figure 7: The density in interval 3)

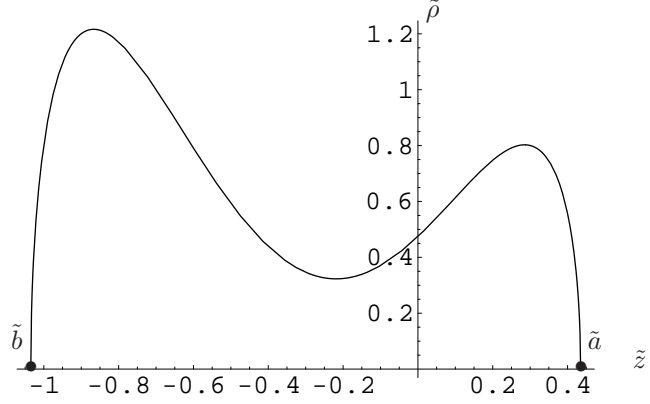


Figure 8: The density in interval 4)

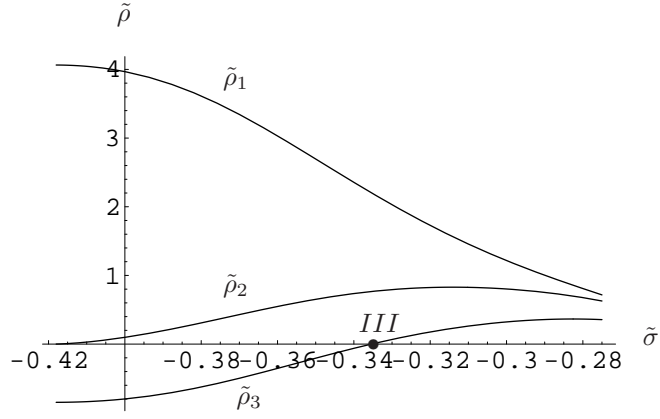


Figure 9: The extremum densities

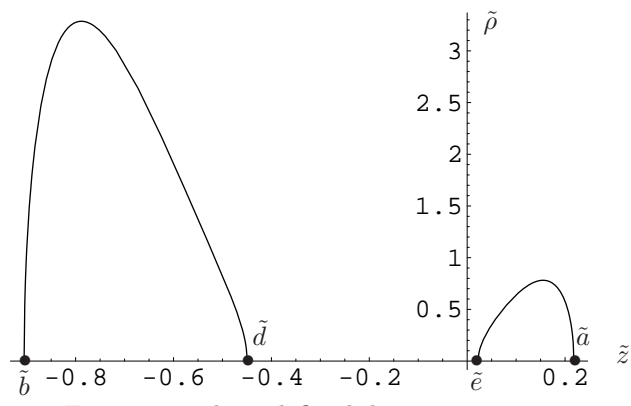


Figure 10: The redefined density

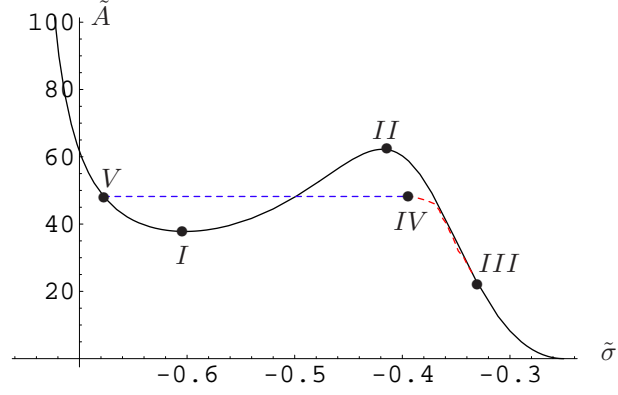


Figure 11: The modified surface area

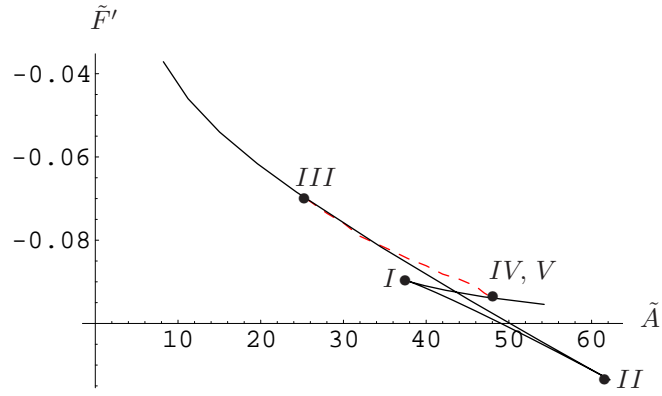


Figure 12: The modified derivative of the free energy

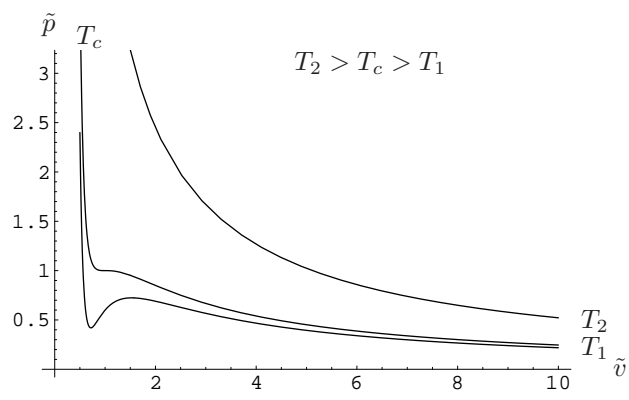


Figure 13: Pressure as a function of volume (isotherms)

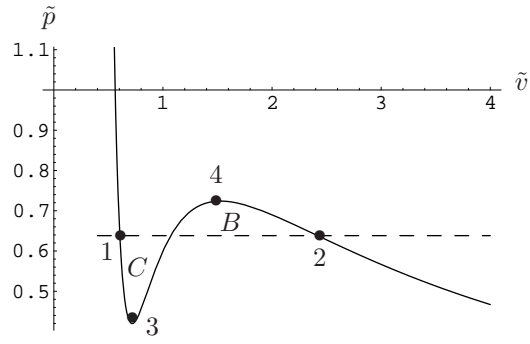


Figure 14: Pressure as a function of volume for $T < T_c$

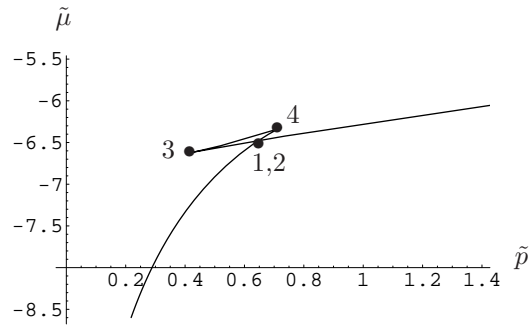


Figure 15: Chemical potential as a function of pressure for $T < T_c$

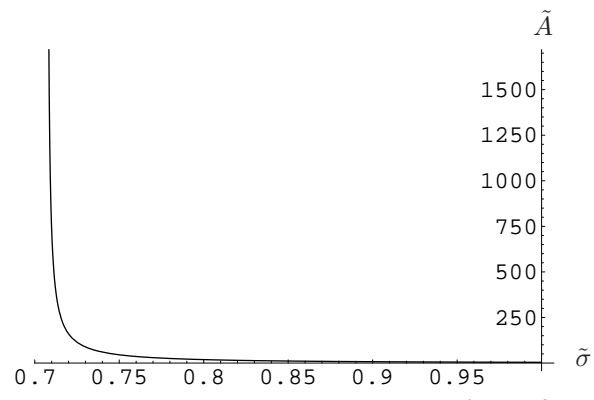


Figure 16: Surface area for $G(z) = z^4 + \lambda z^2$

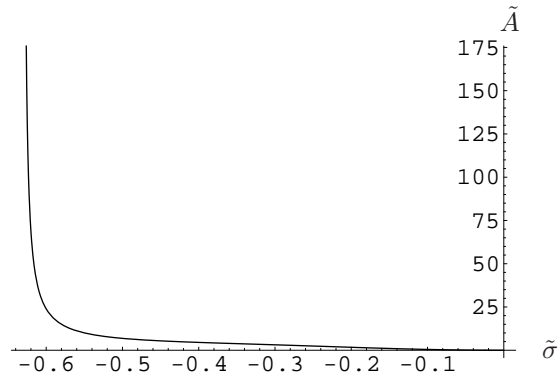


Figure 17: Surface area for $G(z) = z^4 + \lambda z$